Gain of Electromagnetic Horns

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The absolute gain of a standard horn is often measured by determining the transmission loss versus separation between two identical standard horns. Correction ratios are needed because the usual criterion for separation ($2a^2/\lambda$) may not justify the use of the far-zone power transmission formula. Using the near-field power transmission formula, the ratio between the Fraunhofer and Fresnel gain of a pyramidal electromagnetic horn has been computed as a function of horn dimensions and separation distance.

The calculated corrections have been applied in the absolute gain measurement of a standard horn which was used as a calibration reference in a recent 4080-mc gain measurement of a large horn-reflector antenna. The measured gain of the standard horn at 4080 mc is 20.11 db with an accuracy of ±0.035 db. The calculated gain is 20.15 db.

I. INTRODUCTION

Recently, a standard horn was used as a calibration reference in measuring the gain of a 400-square foot aperture horn-reflector antenna at 4080 mcs.[1] Since the horn-reflector antenna is currently being used for precision measurement of the absolute flux of stellar radio sources, it is desirable that the gain of the standard horn be known as accurately as possible. From previous work,[2] the calculated gain[7] of a standard horn was believed to be within ±0.1 db of its true gain. Our purpose was to measure the absolute gain of the standard horn to an accuracy better than that previously achieved.

The gain of a standard horn can be determined by measuring the transmission loss versus separation between two identical standard horns. In the technique of measurement commonly used, the separation distance is not large, and it is well known that the far-zone power transmission formula

$$P_R/P_T = (G\lambda/4\pi r)^2 \tag{1}$$

is not valid if the separation r between the apertures of the two horns

is not great enough. Therefore the gain formula

$$G = \frac{4\pi r}{\lambda} (P_R/P_T)^{\frac{1}{2}} \tag{2}$$

may introduce considerable error when the far-zone gain of pyramidal electromagnetic horns is measured at relatively short distances. Even an aperture-to-aperture separation r of about $2a^2/\lambda$ between two optimum horns, where a is the large dimension of the aperture, introduces an error of the order of 1 db. Jakes[2] suggested the junction of the horn with the feeding waveguide as the reference point for optimum horns. He demonstrated empirically that the error in gain may be reduced to about 0.1 db if r is measured between the reference points of two optimum horns. Braun[3] calculated the error in the gain of electromagnetic horns measured at short distances. However, his assumptions about the received power are questionable, since the power in the transmitted wave was averaged over the receiving aperture. Although the near-field power transmission formula appeared in the literature,[4] to our knowledge it has not been applied to the gain measurement of electromagnetic horns. With the aid of the digital computer, the near-field power transmission formula easily yields the required correction ratios for the far-zone gain of pyramidal electromagnetic horns measured at relatively short distances.

II. CALCULATION OF THE CORRECTIONS

Using the Lorentz reciprocity theorem, it has been shown[4] that the ratio of the received to transmitted power between two antennas at any separation is

$$\frac{P_R}{P_T} = \frac{1/4 \left| \int_S (\mathbf{H}_2 \times \mathbf{E}_1 + \mathbf{E}_2 \times \mathbf{H}_1) \cdot \hat{n}\, ds \right|^2}{\left\{ \mathrm{Re} \int_{s_1} (\mathbf{E}_1 \times \mathbf{H}_1^*) \cdot \hat{n}_1 ds \right\} \left\{ \mathrm{Re} \int_{s_2} (\mathbf{E}_2 \times \mathbf{H}_2^*) \cdot \hat{n}_2 ds \right\}} \tag{3}$$

where $\mathbf{E}_1$, $\mathbf{H}_1$ are the fields when antenna 1 is transmitting, $\mathbf{E}_2$, $\mathbf{H}_2$ are the fields when antenna 2 is transmitting; and $\hat{n}$, $\hat{n}_1$, and $\hat{n}_2$ are the unit normals of the surfaces. The surface S can be either one of the two antenna apertures. Equation (3) is an exact formula if all the field quantities are evaluated with both antennas in place and under matched conditions. In the following calculation the reflections between the antennas will be neglected; that is, in evaluating $\mathbf{E}_1$, $\mathbf{H}_1$ antenna 2 will be removed, and in evaluating $\mathbf{E}_2$, $\mathbf{H}_2$ antenna 1 will be removed. We also neglect any mismatch between antennas and their transmission lines.

Furthermore, we assume that the tangential components of **E** and **H** are related by the free-space impedance at each point:

$$\hat{n} \times \mathbf{E}^t = \sqrt{\frac{\mu}{\epsilon}}\, \mathbf{H}^t.$$

With these approximations, we can write down the power transmission formula between two electromagnetic horns at any separation.

$$\frac{P_R}{P_T} = \frac{\left| \int_{s_1} \int_{s_2} E_1{}^t(P)\, \frac{e^{-jkr}}{r}\, E_2{}^t(P')\, ds\, ds' \right|^2}{\lambda^2 \int_{s_1} |\, E_1{}^t(P)\,|^2 ds \int_{s_2} |\, E_2{}^t(P')\,|^2 ds'} \tag{5}$$

where P and P' are points on the aperture surfaces S_1 and S_2 respectively. Assuming the field at the aperture of the transmitting horn is the same as though the horn were continued (i.e., the usual Kirchhoff approximation), the tangential electric fields in the aperture are given by

$$E_1{}^t = E_1{}^0 \cos \frac{\pi y}{a} \exp - \left[jk \left(\frac{x^2}{2l_E} + \frac{y^2}{2l_H} \right) \right] \tag{6}$$

$$E_2{}^t = E_2{}^0 \cos \frac{\pi \eta}{a} \exp - \left[jk \left(\frac{\zeta^2}{2l_E} + \frac{\eta^2}{2l_H} \right) \right] \tag{7}$$

where l_E and l_H are the E- and H-plane slant heights respectively. The distance r may be approximated by

$$\begin{aligned} r &= [R^2 + (x - \zeta)^2 + (y - \eta)^2]^{\frac{1}{2}} \\ &\approx R + \frac{(x - \zeta)^2 + (y - \eta)^2}{2R}. \end{aligned} \tag{8}$$

All pertinent dimensions are illustrated in Fig. 1. Since the gain measurements usually involve two identical horns, $S_1 = S_2$, and substituting (6), (7), and (8) into (5), (2) reduces to the near-field gain in the Fresnel approximation:

$$G_N = \frac{4\pi \left| \int_s \int_{s'} \cos \frac{\pi y}{a} \cos \frac{\pi \eta}{a} \exp - \left[jk \left\{ \frac{x^2 + \zeta^2}{2l_E} + \frac{y^2 + \eta^2}{2l_H} + \frac{(x - \zeta)^2}{2R} + \frac{(y - \eta)^2}{2R} \right\} \right] ds\, ds' \right|}{\lambda^2 \int_s \cos^2 \frac{\pi y}{a}\, ds} \tag{9}$$

while the Fraunhofer gain is

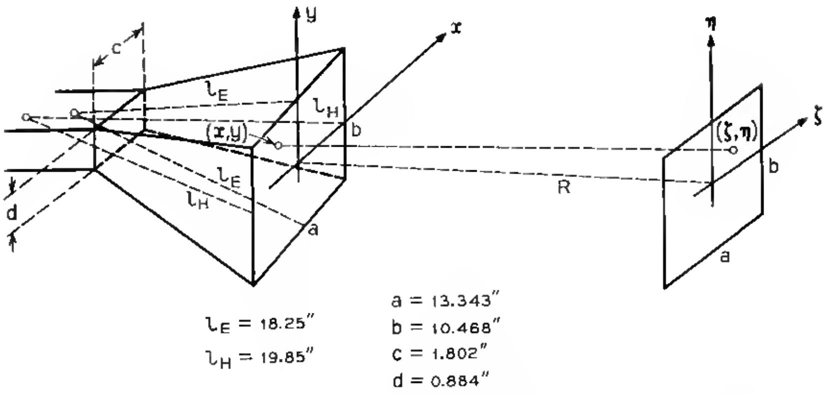

Fig. 1 — Physical dimensions for transmission between two electromagnetic horns.

$$G = \frac{4\pi \left| \int_s \int_{s'} \cos\frac{\pi y}{a} \cos\frac{\pi\eta}{a} \cdot \exp - \left[jk \left(\frac{x^2+\zeta^2}{2l_E} + \frac{y^2+\eta^2}{2l_H} \right) \right] ds\, ds' \right|}{\lambda^2 \int_s \cos^2 \frac{\pi y}{a}\, ds}. \tag{10}$$

Dividing (10) by (9) yields the required correction ratio. It is convenient to split this ratio into the E-plane correction and the H-plane correction

$$C = (G/G_N) = C_E C_H \tag{11}$$

where

$$C_E = \frac{\left| \int_{-b/2}^{b/2} \int_{-b/2}^{b/2} \exp - \left[jk \left(\frac{x^2+\zeta^2}{2l_E} \right) \right] dx\, d\zeta \right|}{\left| \int_{-b/2}^{b/2} \int_{-b/2}^{b/2} \exp - \left[jk \left(\frac{x^2+\zeta^2}{2l_E} \right) \right] \cdot \exp - \left[jk \frac{(x-\zeta)^2}{2R} \right] dx\, d\zeta \right|} \tag{12}$$

and

$$C_H = \frac{\left| \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \cos\frac{\pi y}{a} \cos\frac{\pi\eta}{a} \exp - \left[jk \left(\frac{y^2+\eta^2}{2l_H} \right) \right] dy\, d\eta \right|}{\left| \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \cos\frac{\pi y}{a} \cos\frac{\pi\eta}{a} \exp - \left[jk \left(\frac{y^2+\eta^2}{2l_H} \right) \right] \cdot \exp - \left[jk \frac{(y-\eta)^2}{2R} \right] dy\, d\eta \right|}. \tag{13}$$

The numerators in the above expressions may be identified as Fresnel integrals. After normalizing the parameters, we have

$$C_E = \frac{M\left[C^2\left(\frac{2}{\sqrt{M}}\right) + S^2\left(\frac{2}{\sqrt{M}}\right)\right]}{\left\{\left[\int_{-1}^{1}\int_{-1}^{1} \cos 2\pi \left(\frac{\omega^2 + \zeta^2}{M} + \frac{(\omega - \zeta)^2}{H}\right) d\omega\, d\zeta\right]^2 + \left[\int_{-1}^{1}\int_{-1}^{1} \sin 2\pi \left(\frac{\omega^2 + \zeta^2}{M} + \frac{(\omega - \zeta)^2}{H}\right) d\omega\, d\zeta\right]^2\right\}^{\frac{1}{2}}} \tag{14}$$

and

$$C_H = \frac{\frac{N}{4}\{[C(f) - C(g)]^2 + [S(f) - S(g)]^2\}}{\left\{\left[\int_{-1}^{1}\int_{-1}^{1} \cos\frac{\pi}{2} u \cos\frac{\pi}{2} v \cos 2\pi \left(\frac{u^2 + v^2}{N} + \frac{(u - v)^2}{P}\right) du\, dv\right]^2 + \left[\int_{-1}^{1}\int_{-1}^{1} \cos\frac{\pi}{2} u \cos\frac{\pi}{2} v \sin 2\pi \left(\frac{u^2 + v^2}{N} + \frac{(u - v)^2}{P}\right) du\, dv\right]^2\right\}^{\frac{1}{2}}} \tag{15}$$

where

$$M = 8\lambda l_E/b^2 \qquad H = 8\lambda R/b^2$$

$$N = 8\lambda l_H/a^2 \qquad P = 8\lambda R/a^2$$

$$f = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{N}{8}} + \frac{1}{\sqrt{\frac{N}{8}}}\right) \qquad g = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{N}{8}} - \frac{1}{\sqrt{\frac{N}{8}}}\right).$$

The Fresnel integrals are defined as

$$C(u) = \int_0^u \cos\frac{\pi}{2} t^2\, dt \quad \text{and} \quad S(u) = \int_0^u \sin\frac{\pi}{2} t^2\, dt$$

Equations (14) and (15) have been programmed for a digital computer; the results are summarized in Tables I and II.

It is interesting to notice that there exists substantial discrepancy between our correction ratios and those in Braun's article,[3] especially at short separations. In addition to the approximations made here, Braun employed an averaging process in which the power of the transmitted wave is integrated over the effective receiving aperture area $(\lambda^2/4\pi)G$. Therefore the correction ratios presented here are expected to be much more accurate than Braun's data and they should be useful for precision gain measurement of pyramidal electromagnetic horns.

Table I — E-Plane Corrections (db)

M \ H	8	16	32	64	128	256
2.0	1.740	0.997	0.520	0.263	0.132	0.066
2.5	1.585	0.856	0.426	0.210	0.104	0.051
3.0	1.490	0.757	0.362	0.175	0.085	0.042
3.5	1.418	0.684	0.317	0.150	0.073	0.036
4.0	1.359	0.627	0.284	0.133	0.064	0.031
5.0	1.268	0.547	0.237	0.108	0.051	0.025
6.0	1.201	0.492	0.207	0.092	0.043	0.021
8.0	1.109	0.423	0.168	0.072	0.033	0.016
10.0	1.050	0.381	0.145	0.060	0.027	0.013
32.0	0.870	0.261	0.081	0.028	0.011	0.005
∞	0.779	0.205	0.052	0.010	0.003	0.001

$M = 8\lambda l_E/b^2$ $H = 8\lambda R/b^2$

III. MEASUREMENT TECHNIQUE

The standard horn was mounted in a wooden structure suitably covered with hairflex absorber; a sketch of the horn and its physical dimensions are shown in Fig. 1. A level monorail track was installed along the center line of the floor of an anechoic chamber. A stable, wooden equipment cart was designed to move smoothly along the monorail. One of two identical standard horns with hairflex baffle was mounted on the equipment cart (Fig. 2), the other being mounted in the end wall of the chamber. The equipment set-up is quite conventional and is shown schematically in Fig. 3.

The following procedure was used in the measurements: a reference level was set by removing the standard horns and connecting the waveguides directly (Fig. 3). With the standard horns in place and separated by $r \geqq 2a^2/\lambda$, a series of measurements of received power versus increas-

Table II — H-Plane Corrections (db)

N \ P	8	16	32	64	128	256
2.0	0.833	0.422	0.209	0.104	0.052	0.026
2.5	0.772	0.376	0.181	0.089	0.044	0.022
3.0	0.717	0.336	0.159	0.077	0.038	0.019
3.5	0.671	0.304	0.141	0.067	0.033	0.016
4.0	0.633	0.279	0.127	0.060	0.029	0.014
5.0	0.575	0.242	0.107	0.049	0.024	0.012
6.0	0.533	0.216	0.093	0.042	0.020	0.010
8.0	0.478	0.183	0.075	0.033	0.015	0.007
10.0	0.443	0.162	0.064	0.027	0.013	0.006
32.0	0.340	0.103	0.033	0.012	0.005	0.002
∞	0.291	0.071	0.019	0.005	0.001	0.0002

$N = 8\lambda l_H/a^2$ $P = 8\lambda R/a^2$

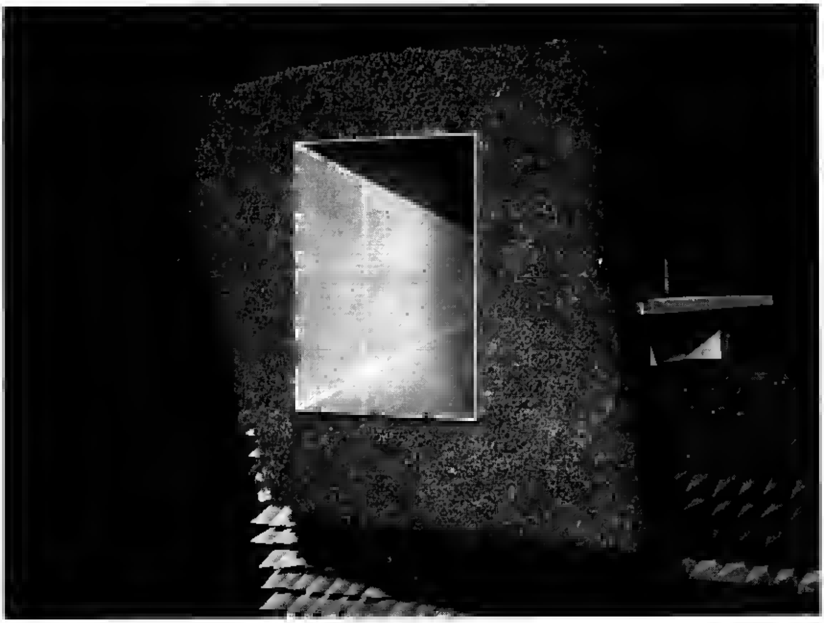

Fig. 2 — Standard horn mounted on equipment cart.

ing (r) were made. After completion of such a series, the reference level was rechecked by removing the standard horns and connecting the waveguides together. The above procedure was repeated several times for vertical and horizontal polarizations.

IV. RESULTS OF MEASUREMENT

The distribution of all of the measured gains at 4080 mc has been plotted as a histogram in Fig. 4. The near-field correction discussed above has been applied to these data. It should be pointed out that occurrences falling on the boundary lines of the columns have been evenly divided between the two neighboring columns; this accounts for the half occurrences which appear in the heights of some of the columns. The mean value of this sample distribution is 20.11 db, and its standard deviation is 0.05 db. The central limit theorem of probability theory indicates a 99.7 per cent confidence interval of $\bar{X} \pm (3\sigma/\sqrt{n})$ for the true mean, where $\bar{X}$ is the sample mean, n is the sample size, and σ is the population standard deviation.[5] Since the present sample size is 90, the population standard deviation should be close to the above sample standard devia-

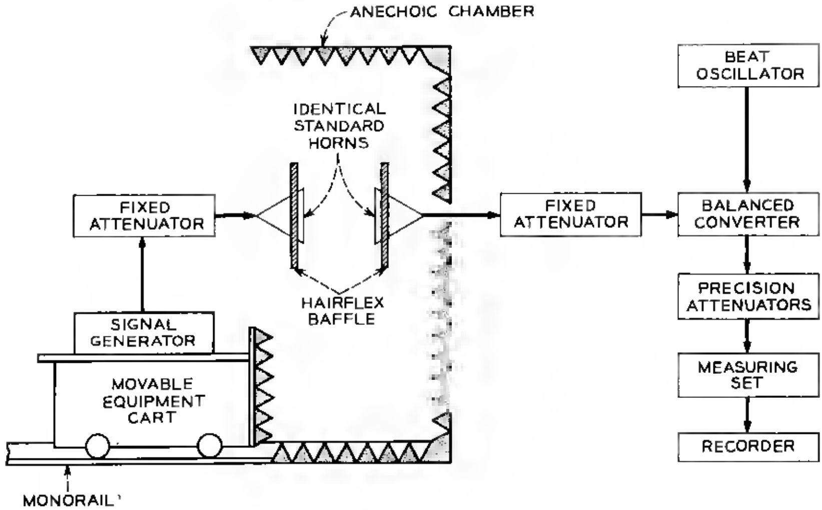

Fig. 3 — Equipment set-up.

tion, 0.05 db; therefore the random error in the mean value 20.11 db is of the order of ± 0.016 db $(3 \times 0.05/\sqrt{90})$.

The spread in the measured gain may be attributed to the following factors:

1.	measuring system stability	± 0.01 db
2.	precision attenuator readings	± 0.015 db
3.	repeatability of electrical connections	± 0.015 db
4.	imperfection of the anechoic chamber	± 0.02 db
5.	interaction between the transmitting horn and the receiving horn	± 0.04 db.

The figures for the above factors are the estimates for one horn; they are half the probable random errors in the transmission between two horn antennas. Half of the measured gains were obtained when the horn apertures were vertically polarized, and half when horizontally polarized; when compared, the difference between the means of the two samples is only 0.01 db. This comparison implies only small errors due to the anechoic chamber.

The interaction effect is clearly demonstrated by the measured $(\lambda/2)$-period oscillation versus separation shown in Figs. 5(a) and 5(b). The amplitude of the oscillation is of the order of 0.05 db, and agrees fairly well with the qualitative calculation of Silver.[6]

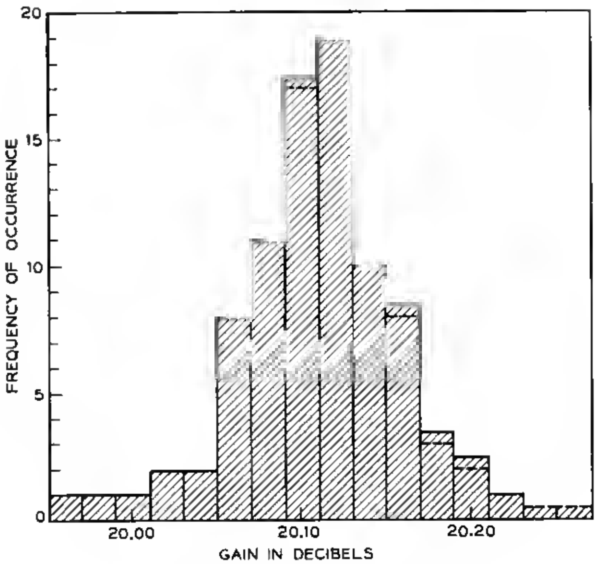

Fig. 4 — Histogram for the measured gains.

In addition to the random errors discussed above, the calibrated precision attenuators hide an absolute error which is constant for all measured gains. The probable value of this error is ± 0.04 db in the power transmission measurement, which contributes ± 0.02 db to the gain error. It follows that the total possible error of the measured gain (which includes the random error and the absolute attenuator error) is about ± 0.035 db. The calculated gain[7] of the standard horn is 20.15 db at 4080 mc. The discrepancy between the calculated value and the measured gain (20.11 db) is 0.04 db.

It should be pointed out that both transmitting and receiving horns in this gain measurement are isolated by 10-db fixed attenuators. However the mismatch at the horn-waveguide junction is not tuned out, because this same mismatch was not tuned out when the standard horn was used as a calibration reference for the gain measurement of the large horn-reflector antenna. A VSWR measurement revealed a reflection coefficient of -25 db, which represents a transmission loss of 0.015 db.

V. SUMMARY AND CONCLUSIONS

Using the near-field power transmission formula, the ratio between the Fraunhofer and Fresnel gain of a pyramidal electromagnetic horn

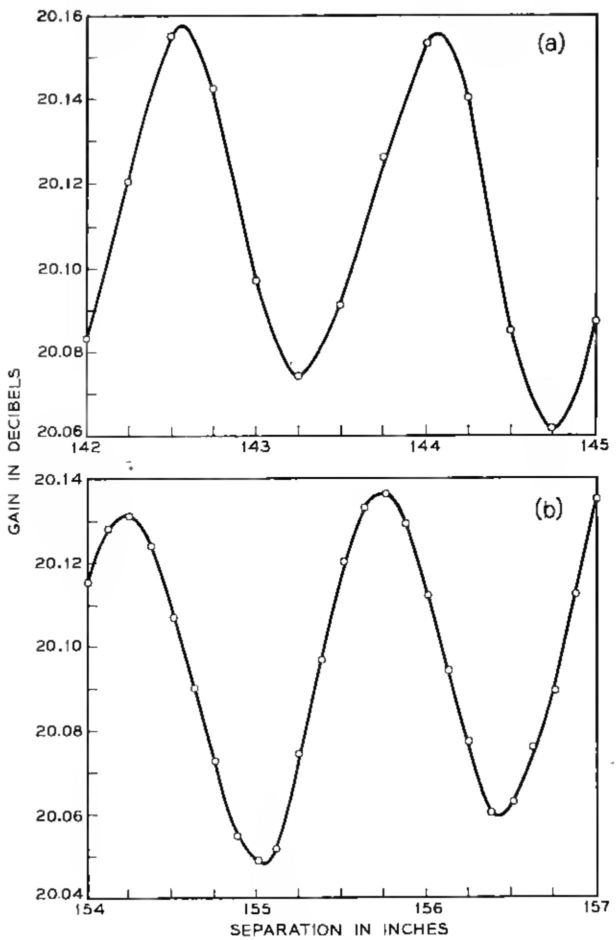

Fig. 5 — Measured gain variation due to interaction.

has been computed as a function of horn dimensions and separation distance. Our computations are expected to be much more accurate than previous data and should be very useful for precision gain measurement of pyramidal electromagnetic horns.

An application of the calculated corrections was made in the absolute gain measurement of a standard horn. The measured gain of the standard horn at 4080 mc is 20.11 db with an accuracy of ± 0.035 db; the calculated gain is 20.15 db. The interaction between two standard horns may introduce an error of the order of 0.05 db in the gain measurement

at a separation distance of $2a^2/\lambda$; however, it is reduced considerably by taking the average of several measurements. The averaging procedure can also reduce other random errors due to environment, measuring system stability, attenuator readings, etc. Using the corrections presented above, together with other careful considerations, it is possible to achieve an accuracy well below 0.1 db in the gain measurement of pyramidal electromagnetic horns.

VI. ACKNOWLEDGMENT

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